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A Non-Differentiable Quantum Variational Embedding in Presence of Time Delays

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Abstract

We develop Cresson's non-differentiable embedding to quantum problems of the calculus of variations and optimal control with time delay. Main results show that the dynamics of non-differentiable Lagrangian and Hamiltonian systems with time delays can be determined, in a coherent way, by the least-action principle.

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1 Introduction

Lagrangian systems play a fundamental role describing motion in mechanics. The importance of such systems is related to the fact that they can be derived via the least-action principle using differentiable manifolds [4]. Nevertheless, some important physical systems involve functions that are non-differentiable.

A non-differentiable calculus was introduced in 1992 by Nottale [13,14]. A rigorous foundation to Nottale's scale-relativity theory was recently given by Cresson [5, 6, 8]. The calculus of variations developed in [8] cover sets of non-differentiable curves, by substituting the classical derivative by a new complex operator, known as the *scale derivative*. In [2, 3] Almeida and Torres obtain several Euler–Lagrange equations for variational functionals and isoperimetric problems of the calculus of variations defined on a set of Hölder curves.

The embedding procedure, introduced for stochastic processes in [7], can always be applied to Lagrangian systems [9, 15]. In this work we prove that the embedding

of Lagrangian and Hamiltonian systems with time delays, via the least-action principle, respect the principle of coherence. For the importance of variational and control systems with delays we refer the reader to [10] and references therein.

The article is organized as follows. A brief review of the quantum calculus of [8], which extends the classical differential calculus to non-differentiable functions, is given in Section 2. In Section 3 we discuss the non-differentiable embedding within the time delay formalism: Section 3.1 is devoted to the development of the non-differentiable embedding to variational problems with time delay, where a causal and coherent embedding is obtained by restricting the set of variations; in Section 3.2 we prove that the non-differentiable embedding of the corresponding Hamiltonian formalism is also coherent.

2 The Quantum Calculus of Cresson

Let $\mathbb{X}^d$ denote the set $\mathbb{R}^d$ or $\mathbb{C}^d$, $d \in \mathbb{N}$, and I be an open set in $\mathbb{R}$ with $[t_1, t_2] \subset I$, $t_1 < t_2$. We denote by $\mathcal{G}\left(I, \mathbb{X}^d\right)$ the set of functions $f : I \to \mathbb{X}^d$ and by $\mathcal{C}^0\left(I, \mathbb{X}^d\right)$ the subset of functions of $\mathcal{G}\left(I, \mathbb{X}^d\right)$ that are continuous.

Definition 2.1 (The ϵ-left and ϵ-right quantum derivatives)**.** Let $f \in \mathcal{C}^0\left(I, \mathbb{R}^d\right)$. For all $\epsilon > 0$, the ϵ-left and ϵ-right quantum derivatives of f, denoted respectively by $\Delta_\epsilon^- f$ and $\Delta_\epsilon^+ f$, are defined by

$$\Delta_\epsilon^- f(t) = \frac{f(t) - f(t-\epsilon)}{\epsilon} \quad \text{and} \quad \Delta_\epsilon^+ f(t) = \frac{f(t+\epsilon) - f(t)}{\epsilon}\,.$$

Remark 2.2. The ϵ-left and ϵ-right quantum derivatives of a continuous function f correspond to the classical derivative of the ϵ-mean function f_ϵ^σ defined by

$$f_\epsilon^\sigma(t) = \frac{\sigma}{\epsilon}\int_t^{t+\sigma\epsilon} f(s)ds\,, \quad \sigma = \pm\,.$$

Next we define an operator which generalize the classical derivative.

Definition 2.3 (The ϵ-scale derivative)**.** Let $f \in \mathcal{C}^0\left(I, \mathbb{R}^d\right)$. For all $\epsilon > 0$, the ϵ-scale derivative of f, denoted by $\frac{\Box_\epsilon f}{\Box t}$, is defined by

$$\frac{\Box_\epsilon f}{\Box t} = \frac{1}{2}\left[\left(\Delta_\epsilon^+ f + \Delta_\epsilon^- f\right) - i\left(\Delta_\epsilon^+ f - \Delta_\epsilon^- f\right)\right],$$

where i is the imaginary unit.

Remark 2.4. If f is differentiable, we can take the limit of the scale derivative when ϵ goes to zero. We then obtain the classical derivative $\frac{df}{dt}$ of f.

We also need to extend the scale derivative to complex valued functions.

Definition 2.5. Let $f \in \mathcal{C}^0\left(I, \mathbb{C}^d\right)$ be a continuous complex valued function. For all $\epsilon > 0$, the ϵ scale derivative of f, denoted by $\frac{\Box_\epsilon f}{\Box t}$, is defined by

$$\frac{\Box_\epsilon f}{\Box t} = \frac{\Box_\epsilon \mathrm{Re}(f)}{\Box t} + i\frac{\Box_\epsilon \mathrm{Im}(f)}{\Box t}\,,$$

where $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ denote the real and imaginary part of f, respectively.

In Definition 2.3, the ϵ-scale derivative depends on ϵ, which is a free parameter related to the smoothing order of the function. This brings many difficulties in applications to Physics, when one is interested in particular equations that do not depend on an extra parameter. To solve these problems, the authors of [8] introduced a procedure to extract information independent of ϵ but related with the mean behavior of the function.

Definition 2.6. Let $C^0_{conv}\left(I \times (0,1), \mathbb{R}^d\right) \subseteq C^0\left(I \times (0,1), \mathbb{R}^d\right)$ be such that for any function $f \in C^0_{conv}\left(I \times (0,1), \mathbb{R}^d\right)$ the $\lim\limits_{\epsilon \to 0} f(t,\epsilon)$ exists for any $t \in I$; and E be a complementary of $C^0_{conv}\left(I \times (0,1), \mathbb{R}^d\right)$ in $C^0\left(I \times (0,1), \mathbb{R}^d\right)$. We define the projection map π by

$$\begin{array}{rccl} \pi: & C^0_{conv}\left(I \times (0,1), \mathbb{R}^d\right) \oplus E & \to & C^0_{conv}(I \times \left(0,1), \mathbb{R}^d\right) \\ & f_{conv} + f_E & \mapsto & f_{conv} \end{array}$$

and the operator $\langle \cdot \rangle$ by

$$\begin{array}{rccl} \langle \cdot \rangle: & C^0\left(I \times (0,1), \mathbb{R}^d\right) & \to & C^0\left(I, \mathbb{R}^d\right) \\ & f & \mapsto & \langle f \rangle : t \mapsto \lim\limits_{\epsilon \to 0} \pi(f)(t,\epsilon)\,. \end{array}$$

We now introduce the quantum derivative of f without the dependence of ϵ [8].

Definition 2.7. The quantum derivative of f in the space $\mathcal{C}^0\left(I, \mathbb{R}^d\right)$ is given by the rule

$$\frac{\Box f}{\Box t} = \left\langle \frac{\Box_\epsilon f}{\Box t} \right\rangle. \tag{2.1}$$

The scale derivative (2.1) has some nice properties. Namely, it satisfies a Leibniz and a Barrow rule. First let us recall the definition of an α-Hölderian function.

Definition 2.8 (Hölderian function of exponent α)**.** Let $f \in C^0\left(I, \mathbb{R}^d\right)$. We say that f is α-Hölderian, $0 < \alpha < 1$, if for all $\epsilon > 0$ and all t, $t' \in I$ there exists a constant $c > 0$ such that $|t - t'| \leqslant \epsilon$ implies $\|f(t) - f(t')\| \leqslant c\epsilon^\alpha$, where $\|\cdot\|$ is a norm in $\mathbb{R}^d$. The set of Hölderian functions of Hölder exponent α is denoted by $H^\alpha(I, \mathbb{R}^d)$.

In what follows, we will frequently use $\Box$ to denote the scale derivative operator $\frac{\Box}{\Box t}$.

Theorem 2.9 (The quantum Leibniz rule [8]). *For $f \in H^{\alpha}\left(I, \mathbb{R}^{d}\right)$ and $g \in H^{\beta}\left(I, \mathbb{R}^{d}\right)$, with $\alpha+\beta>1$, one has*

$$\Box(f \cdot g)(t)=\Box f(t) \cdot g(t)+f(t) \cdot \Box g(t) . \tag{2.2}$$

Remark 2.10. For $f \in \mathcal{C}^{1}\left(I, \mathbb{R}^{d}\right)$ and $g \in \mathcal{C}^{1}\left(I, \mathbb{R}^{d}\right)$, one obtains from (2.2) the classical Leibniz rule $(f \cdot g)^{\prime}=f^{\prime} \cdot g+f \cdot g^{\prime}$. For convenience of notation, we sometimes write (2.2) as $(f \cdot g)^{\Box}(t)=f^{\Box}(t) \cdot g(t)+f(t) \cdot g^{\Box}(t)$.

Theorem 2.11 (The quantum Barrow rule [8]). *Let $f \in \mathcal{C}^{0}([t_1, t_2], \mathbb{R})$ be such that $\Box f / \Box t$ is continuous and*

$$\lim_{\epsilon \to 0} \int_{t_1}^{t_2}\left(\frac{\Box_{\epsilon} f}{\Box t}\right)_{E}(t) dt=0. \tag{2.3}$$

Then,

$$\int_{t_1}^{t_2} \frac{\Box f}{\Box t}(t)\, dt=f(t_2)-f(t_1).$$

3 The Non-Differentiable Embedding with Time Delays

Given two operators A and B, we use the notations

$$(A \cdot B)(y)=A(y) B(y) \quad \text{and } (A \circ B)(y)=A(B(y)).$$

Definition 3.1 (The kth scale derivative). Let $k \in \mathbb{N}$. The operator $\Box^{k}$ is defined by

$$\Box^{k}=\frac{\Box^{k}}{\Box t^{k}}=\frac{\Box}{\Box t} \circ \cdots \circ \frac{\Box}{\Box t}, \tag{3.1}$$

where $\dfrac{\Box}{\Box t}$ appears exactly k times on the right-hand side of (3.1).

Definition 3.2 (The backward shift operator ρ^{τ}). Given $\tau>0$, the backward shift operator ρ^{τ} is defined by $\rho^{\tau}(t)=t-\tau$.

Definition 3.3 (The operators $[\cdot]_{\tau}^{\Box^{k}}$ and $[\cdot]_{\tau}^{k}$). Let $\tau>0$, $k \in \mathbb{N}$. For convenience, we introduce the operators $[\cdot]_{\tau}^{\Box^{k}}$ and $[\cdot]_{\tau}^{k}$ by

$$[y]_{\tau}^{\Box^{k}}(t)=\left(t, y(t), \Box y(t), \ldots, \Box^{k} y(t),(y \circ \rho^{\tau})(t),(\Box y \circ \rho^{\tau})(t), \ldots,(\Box^{k} y \circ \rho^{\tau})(t)\right)$$

and

$$[y]_{\tau}^{k}(t)=\left(t, y(t), y^{\prime}(t), \ldots, y^{(k)}(t),(y \circ \rho^{\tau})(t),(y^{\prime} \circ \rho^{\tau})(t), \ldots,(y^{(k)} \circ \rho^{\tau})(t)\right).$$

When $k=1$, we omit k and use $\dot{y}$ to denote the derivative $y^{\prime}$, that is:

$$[y]_{\tau}^{\Box}(t)=(t, y(t), \Box y(t), y(t-\tau), \Box y(t-\tau))$$

and

$$[y]_{\tau}(t)=(t, y(t), \dot{y}(t), y(t-\tau), \dot{y}(t-\tau)).$$

Given a function $f : \mathbb{R} \times \left(\mathbb{C}^d\right)^{2(k+1)} \to \mathbb{C}$, we denote by $F^{k,\tau}$ the corresponding *evaluation operator* defined by $F^{k,\tau} = f[\cdot]_\tau^k$, that is,

$$F^{k,\tau} : \begin{array}{ccc} \mathcal{C}^0\left(I, \mathbb{C}^d\right) & \longrightarrow & \mathcal{C}^0\left(I, \mathbb{C}\right) \\ y & \longmapsto & t \mapsto f[y]_\tau^k(t) \,. \end{array}$$

Let $\mathbf{f} = \{f_i\}_{i=0,\ldots,n}$ and $\mathbf{g} = \{g_i\}_{i=0,\ldots,n}$ be a finite family of functions $f_i, g_i : \mathbb{R} \times \left(\mathbb{C}^d\right)^{2(k+1)} \to \mathbb{C}$, and $F_i^{k,\tau}$ and $G_i^{k,\tau}$, $i = 1, \ldots, n$, be the corresponding family of evaluation operators. We denote by $\mathrm{O}_{\mathbf{f},\mathbf{g}}^{k,\tau}$ the differential operator

$$\mathrm{O}_{\mathbf{f},\mathbf{g}}^{k,\tau} = \sum_{i=0}^{n} F_i^{k,\tau} \cdot \left(\frac{d^i}{dt^i} \circ G_i^{k,\tau}\right), \tag{3.2}$$

with the convention that $\left(\dfrac{d}{dt}\right)^0$ is the identity mapping on $\mathbb{C}$. As before, we omit k when $k = 1$: $\mathrm{O}_{\mathbf{f},\mathbf{g}}^{\tau} = \mathrm{O}_{\mathbf{f},\mathbf{g}}^{1,\tau}$.

Definition 3.4 (Non-differentiable embedding of operators with time delay)**.** The non-differentiable embedding of (3.2), denoted by $\mathrm{Emb}_\Box\left(\mathrm{O}_{\mathbf{f},\mathbf{g}}^{k,\tau}\right)$, is given by

$$\mathrm{Emb}_\Box\left(\mathrm{O}_{\mathbf{f},\mathbf{g}}^{k,\tau}\right) = \sum_{i=0}^{n} F_i^{\Box^k,\tau} \cdot \left(\frac{\Box^i}{\Box t^i} \circ G_i^{\Box^k,\tau}\right),$$

$$F_i^{\Box^k,\tau} = \mathrm{Emb}_\Box(F_i^{k,\tau}) = f_i[\cdot]_\tau^{\Box^k},\ \frac{\Box^i}{\Box t^i} = \mathrm{Emb}_\Box\left(\frac{d^i}{dt^i}\right),\ G_i^{\Box^k,\tau} = \mathrm{Emb}_\Box(G_i^{k,\tau}) = g_i[\cdot]_\tau^{\Box^k}.$$

3.1 Embedding of Variational Problems with Time Delay

The fundamental problem of the calculus of variations with time delay is to minimize

$$I^\tau[q] = \int_{t_1}^{t_2} L\left(t, q(t), \dot{q}(t), q(t-\tau), \dot{q}(t-\tau)\right) dt \tag{3.3}$$

subject to $q(t) = \delta(t)$, $t \in [t_1 - \tau, t_1]$, and $q(t_2) = q_2$, where $t_1 < t_2$ are fixed in $\mathbb{R}$, τ is a given positive real number such that $\tau < t_2 - t_1$, δ is a given piecewise smooth function, and $q_2 \in \mathbb{R}^d$. We assume that admissible functions q are such that both q and $q \circ \rho^\tau$ belong to $\mathcal{C}^1\left(I, \mathbb{R}^d\right)$. Note that, with our notations, (3.3) can be written as

$$I^\tau[q] = \int_{t_1}^{t_2} L[q]_\tau(t) dt.$$

A variation of $q \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$ is another function of $\mathcal{C}^1\left(I, \mathbb{R}^d\right)$ of the form $q + \varepsilon h$ with ε a small number and $h \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$ such that $h(t) = 0$ for $t \in [t_1 - \tau, t_1] \cup \{t_2\}$.

Definition 3.5 (Extremal)**.** We say that q is an *extremal* of funcional (3.3) if

$$\frac{d}{d\varepsilon} I^{\tau}[y+\varepsilon h]|_{\varepsilon=0} = 0$$

for any $h \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$.

A first idea to obtain a non-differentiable Lagrangian system with time delays is to embed directly the classical Euler–Lagrange equations with time delays.

Theorem 3.6 (Euler–Lagrange equations with time delay [1,12])**.** *A function* $q \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$ *is an extremal of* (3.3) *if and only if*

$$\begin{cases} \frac{d}{dt}\left[L_{\dot{q}}[q]_\tau(t) + L_{\dot{q}_\tau}[q]_\tau(t+\tau)\right] \\ \qquad = L_q[q]_\tau(t) + L_{q_\tau}[q]_\tau(t+\tau)\,, \quad t_1 \le t \le t_2-\tau, \\ \frac{d}{dt} L_{\dot{q}}[q]_\tau(t) = L_q[q]_\tau(t)\,, \quad t_2-\tau \le t \le t_2, \end{cases} \tag{EL}$$

where $L_\xi(t,q,\dot{q},q_\tau,\dot{q}_\tau)$ *denotes the partial derivative of* $L(t,q,\dot{q},q_\tau,\dot{q}_\tau)$ *with respect to* $\xi \in \{q,\dot{q},q_\tau,\dot{q}_\tau\}$.

The following theorem gives the non-differentiable embedding of the Euler–Lagrange equations with time delay (EL). By $\mathcal{C}^n_\Box\left(I, \mathbb{X}^d\right)$ we denote the set of functions q such that both q and $q \circ \rho^\tau$ belong to $\mathcal{C}^0\left(I, \mathbb{X}^d\right)$ as well as $\Box^i q$ and $(\Box^i q)\circ\rho^\tau$ for all $i = 1,\ldots,n$.

Theorem 3.7. *Let the Lagrangian* L *be a* $\mathcal{C}^1_\Box\left(I, \mathbb{R}^d\right)$*-function with respect to all its arguments, holomorphic with respect to* $\dot{q}(t)$ *and* $\dot{q}(t-\tau)$*, and real when* $\dot{q}(t)$ *and* $\dot{q}(t-\tau)$ *belong to* $\mathbb{R}^d$*. The non-differentiable embedded* Euler–Lagrange equations with time delay *associated to* L *are given by*

$$\begin{cases} \frac{\Box}{\Box t}\left[L_{\Box q}[q]^\Box_\tau(t) + L_{\Box q_\tau}[q]^\Box_\tau(t+\tau)\right] \\ \qquad = L_q[q]^\Box_\tau(t) + L_{q_\tau}[q]^\Box_\tau(t+\tau), \;\; t_1 \le t \le t_2-\tau, \\ \frac{\Box}{\Box t} L_{\Box q}[q]^\Box_\tau(t) = L_q[q]^\Box_\tau(t), \;\; t_2-\tau \le t \le t_2\,. \end{cases} \tag{$\Box EL$}$$

Proof. The Euler–Lagrange equations (EL) can be written in the equivalent form

$$\mathrm{O}^\tau_{\mathbf{f},\mathbf{g}}(q)(t) = 0, \quad t \in [t_1, t_2],$$

with $\mathbf{f}$ and $\mathbf{g}$ given by

$$\mathbf{f}[q]_\tau(t) = \begin{cases} (-L_q[q]_\tau(t) - L_{q_\tau}[q]_\tau(t+\tau), 1) & \text{if } t \in [t_1, t_2-\tau] \\ (-L_q[q]_\tau(t), 1) & \text{if } t \in [t_2-\tau, t_2] \end{cases}$$

and

$$\mathbf{g}[q]_\tau(t) = \begin{cases} (1, L_{\dot{q}}[q]_\tau(t) + L_{\dot{q}_\tau}[q]_\tau(t+\tau)) & \text{if } t \in [t_1, t_2-\tau] \\ (1, L_{\dot{q}}[q]_\tau(t)) & \text{if } t \in [t_2-\tau, t_2]. \end{cases}$$

The intended conclusion follows now by a direct application of Definition 3.4. □

Remark 3.8. The Euler–Lagrange equations ($\Box EL$) reduce to (EL) when the functions are differentiable.

Another approach to obtain a non-differentiable Lagrangian system with time delays is to embed the Lagrangian, and then to apply the least-action principle. The non-differentiable embedding of functional (3.3) is given by

$$I_{\Box}^{\tau}[q] = \int_{t_1}^{t_2} L\left(t, q(t), \Box q(t), q(t-\tau), \Box q(t-\tau)\right) dt = \int_{t_1}^{t_2} L[y]_{\tau}^{\Box}(t) dt. \tag{3.4}$$

In contrast with the original problem, the admissible functions q are now not necessarily differentiable: admissible functions q for (3.4) are those such that $q \in \mathcal{C}_{\Box}^{1}\left(I, \mathbb{R}^d\right)$.

Let $\alpha, \beta, \varepsilon \in \mathbb{R}$ be such that $0 < \alpha, \beta < 1$, $\alpha + \beta > 1$ and $|\varepsilon| \ll 1$. A variation of $q \in H^{\alpha}\left(I, \mathbb{R}^d\right)$ is another function of $H^{\alpha}\left(I, \mathbb{R}^d\right)$ of the form $q + \varepsilon h$, with $h \in H^{\beta}\left(I, \mathbb{R}^d\right)$, such that $h(t) = 0$ for $t \in [t_1 - \tau, t_1] \cup \{t_2\}$.

Definition 3.9 (Non-differentiable extremal)**.** We say that q is a *non-differentiable extremal* of funcional (3.4) if

$$\frac{d}{d\varepsilon} I_{\Box}^{\tau}[y + \varepsilon h]\Big|_{\varepsilon=0} = 0 \tag{3.5}$$

for any $h \in H^{\beta}\left(I, \mathbb{R}^d\right)$.

As in the classical case, the least-action principle has here its own meaning, i.e., we seek the non-differentiable extremals of funcional (3.4) to determine the dynamics of a non-differentiable dynamical system. The next theorem gives the Euler–Lagrange equations with time delay obtained from the non-differentiable least-action principle.

Theorem 3.10. *Let* $0 < \alpha, \beta < 1$ *with* $\alpha + \beta > 1$. *If* $q \in H^{\alpha}\left(I, \mathbb{R}^d\right)$ *satisfies* $\Box q \in H^{\alpha}\left(I, \mathbb{R}^d\right)$ *and* $\left(L_{\Box q}[q]_{\tau}^{\Box}(t) + L_{\Box q_{\tau}}[q]_{\tau}^{\Box}(t)\right) \cdot h(t)$ *satisfies condition* (2.3) *for all* $h \in H^{\beta}\left(I, \mathbb{R}^d\right)$, *then function* q *is a non-differentiable extremal of* $I_{\Box}^{\tau}$ *if and only if*

$$\begin{cases} \dfrac{\Box}{\Box t}\left(L_{\Box q}[q]_{\tau}^{\Box}(t) + L_{\Box q_{\tau}}[q]_{\tau}^{\Box}(t+\tau)\right) \\ \qquad\qquad = L_{q}[q]_{\tau}^{\Box}(t) + L_{q_{\tau}}[q]_{\tau}^{\Box}(t+\tau), \quad t_1 \leq t \leq t_2 - \tau, \\ \dfrac{\Box}{\Box t} L_{\Box q}[q]_{\tau}^{\Box}(t) = L_{q}[q]_{\tau}^{\Box}(t), \quad t_2 - \tau \leq t \leq t_2. \end{cases} \tag{$EL_{\Box LAP}$}$$

Proof. Condition (3.5) gives

$$\int_{t_1}^{t_2} \left(L_{q}[q]_{\tau}^{\Box}(t) \cdot h(t) + L_{\Box q}[q]_{\tau}^{\Box}(t) \cdot \Box h(t)\right) dt \\ + \int_{t_1}^{t_2} \left(L_{q_{\tau}}[q]_{\tau}^{\Box}(t) \cdot h(t-\tau) + L_{\Box q_{\tau}}[q]_{\tau}^{\Box}(t) \cdot \Box h(t-\tau)\right) dt = 0\,. \tag{3.6}$$

By the linear change of variables $t = s + \tau$ in the last integral of (3.6), and having in mind the fact that $h(t) = 0$ on $[t_1 - \tau, t_1]$, equation (3.6) becomes

$$\begin{aligned}\int_{t_1}^{t_2-\tau} &\left[\left(L_q[q]_\tau^\Box(t) + L_{q_\tau}[q]_\tau^\Box(t+\tau)\right)\cdot h(t)\right.\\ &\qquad \left.+\left(L_{\Box q}[q]_\tau^\Box(t) + L_{\Box q_\tau}[q]_\tau^\Box(t+\tau)\right)\cdot \Box h(t)\right] dt\\ &\qquad\qquad + \int_{t_2-\tau}^{t_2}\left(L_q[q]_\tau^\Box(t)\cdot h(t) + L_{\Box q}[q]_\tau^\Box(t)\cdot\Box h(t)\right) dt = 0\,. \qquad (3.7)\end{aligned}$$

Using the hypotheses of the theorem, we obtain from Theorem 2.9 that

$$\begin{aligned}\int_{t_1}^{t_2-\tau} &\left(L_{\Box q}[q]_\tau^\Box(t) + L_{\Box q_\tau}[q]_\tau^\Box(t+\tau)\right)\cdot \Box h(t)dt\\ &= \int_{t_1}^{t_2-\tau}\frac{\Box}{\Box t}\left\{\left(L_{\Box q}[q]_\tau^\Box(t) + L_{\Box q_\tau}[q]_\tau^\Box(t+\tau)\right)\cdot h(t)\right\} dt\\ &\qquad - \int_{t_1}^{t_2-\tau}\frac{\Box}{\Box t}\left(L_{\Box q}[q]_\tau^\Box(t) + L_{\Box q_\tau}[q]_\tau^\Box(t+\tau)\right)\cdot h(t)dt \qquad (3.8)\end{aligned}$$

and

$$\begin{aligned}\int_{t_2-\tau}^{t_2} &L_{\Box q}[q]_\tau^\Box(t)\cdot\Box h(t)dt\\ &= \int_{t_2-\tau}^{t_2}\frac{\Box}{\Box t}\left(L_{\Box q}[q]_\tau^\Box(t)\cdot h(t)\right)dt - \int_{t_2-\tau}^{t_2}\frac{\Box}{\Box t}\left(L_{\Box q}[q]_\tau^\Box(t)\right)\cdot h(t)dt\,. \qquad (3.9)\end{aligned}$$

Because $\left(L_{\Box q}[q]_\tau^\Box(t) + L_{\Box q_\tau}[q]_\tau^\Box(t+\tau)\right)\cdot h(t)$ satisfies (2.3) for all $h \in H^\beta\left(I, \mathbb{R}^d\right)$, using Theorem 2.11 and replacing the quantities of (3.8) and (3.9) into (3.7), we obtain

$$\begin{aligned}\int_{t_1}^{t_2-\tau} &\left[L_q[q]_\tau^\Box(t) + L_{q_\tau}[q]_\tau^\Box(t+\tau) - \frac{\Box}{\Box t}\left(L_{\Box q}[q]_\tau^\Box(t) + L_{\Box q_\tau}[q]_\tau^\Box(t+\tau)\right)\right]\cdot h(t)dt\\ &+ \left(L_{\Box q}[q]_\tau^\Box(t) + L_{\Box q_\tau}[q]_\tau^\Box(t+\tau)\right)\cdot h(t)\Big|_{t_1}^{t_2-\tau}\\ &+ \int_{t_2-\tau}^{t_2}\left[L_q[q]_\tau^\Box(t) - \frac{\Box}{\Box t}\left(L_{\Box q}[q]_\tau^\Box(t)\right)\right]\cdot h(t)dt + L_{\Box q}[q]_\tau^\Box(t)\cdot h(t)\Big|_{t_2-\tau}^{t_2} = 0\,.\end{aligned}$$

The Euler–Lagrange equations with time delay ($EL_{\Box LAP}$) are obtained using the fundamental lemma of the calculus of variations (see, e.g., [11]). □

To summarize, the dynamics of a non-differentiable Lagrangian system with time delay are determined by the Euler–Lagrange equations ($EL_{\Box LAP}$), via the □-least-action principle, respecting the principle of coherence: ($EL_{\Box LAP}$) coincide with ($\Box EL$).

3.2 Embedding of Optimal Control Problems with Time Delay

In Section 3.1 we studied the non-differentiable variational embedding in presence of time delays. Now, we give a scale Hamiltonian embedding for more general scale problems of optimal control with delay time. Following [10, 12], we define the optimal control problem with time delay as follows:

$$I^{\tau}[q,u] = \int_{t_1}^{t_2} L\left(t, q(t), q(t-\tau), u(t), u(t-\tau)\right) dt \longrightarrow \min\,, \tag{3.10}$$

$$\dot{q}(t) = \varphi\left(t, q(t), q(t-\tau), u(t), u(t-\tau)\right)\,, \tag{3.11}$$

under given boundary conditions

$$q(t) = \delta(t), \quad t \in [t_1 - \tau, t_1]\,, \quad q(t_2) = q_2, \tag{3.12}$$

where $q \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$, $u \in \mathcal{C}^0\left(I, \mathbb{R}^d\right)$, the Lagrangian $L : I \times \mathbb{R}^{2d} \times \mathbb{R}^{2m} \to \mathbb{R}$ and the velocity vector $\varphi : I \times \mathbb{R}^{2d} \times \mathbb{R}^{2m} \to \mathbb{R}^d$ are assumed to be C^1-functions with respect to all its arguments. Similarly as before, we assume that δ is a given piecewise smooth function and q_2 a given vector in $\mathbb{R}^d$.

Definition 3.11. We introduce the operators $[\cdot,\cdot]_\tau$, $[\cdot,\cdot,\cdot]_\tau$, $[\cdot,\cdot]_\tau^\square$ and $[\cdot,\cdot]_\tau^\square$ by:

1. $[q,u]_\tau(t) = (t, q(t), q(t-\tau), u(t), u(t-\tau))$, where $q \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$ and $u \in \mathcal{C}^0\left(I, \mathbb{R}^d\right)$;
2. $[q,u,p]_\tau(t) = (t, q(t), q(t-\tau), u(t), u(t-\tau), p(t))$, where $q, p \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$ and $u \in \mathcal{C}^0\left(I, \mathbb{R}^d\right)$;
3. $[q,u]_\tau^\square(t) = (t, q(t), q(t-\tau), u(t), u(t-\tau))$, where $q \in H^\alpha\left(I, \mathbb{R}^d\right)$ and $u \in H^\alpha\left(I, \mathbb{C}^d\right)$;
4. $[q,u,p]_\tau^\square(t) = (t, q(t), q(t-\tau), u(t), u(t-\tau), p(t))$, where $q \in H^\alpha\left(I, \mathbb{R}^d\right)$ and $u, p \in H^\alpha\left(I, \mathbb{C}^d\right)$.

Theorem 3.12 ([10, 12]). *If (q,u) is a minimizer to problem* (3.10)–(3.12)*, then there exists a co-vector function $p \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$ such that the following conditions hold:*

- the Hamiltonian systems

$$\begin{cases} \dot{q}(t) = H_p[q,u,p]_\tau(t)\,, \\ \dot{p}(t) = -H_q[q,u,p]_\tau(t) - H_{q_\tau}[q,u,p]_\tau(t+\tau)\,, \end{cases} \tag{3.13}$$

for $t_1 \le t \le t_2 - \tau$, and

$$\begin{cases} \dot{q}(t) = H_p[q,u,p]_\tau(t)\,, \\ \dot{p}(t) = -H_q[q,u,p]_\tau(t)\,, \end{cases} \tag{3.14}$$

for $t_2 - \tau \le t \le t_2$;

- the stationary conditions

$$H_u[q,u,p]_\tau(t) + H_{u_\tau}[q,u,p]_\tau(t+\tau) = 0\,, \tag{3.15}$$

for $t_1 \le t \le t_2 - \tau$, *and*

$$H_u[q,u,p]_\tau(t) = 0, \tag{3.16}$$

for $t_2 - \tau \le t \le t_2$;

where the Hamiltonian H *is defined by* $H[q,u,p]_\tau(t) = L[q,u]_\tau(t) + p(t)\cdot\varphi[q,u]_\tau(t)$.

Lemma 3.13. *Let* $H[q,u,p]_\tau^\Box(t) = L[q,u]_\tau^\Box(t) + p(t)\cdot\varphi[q,u]_\tau^\Box(t)$. *The embedding of the Hamiltonian systems* (3.13) *and* (3.14) *are given, respectively, by*

$$\begin{cases} \Box q(t) = H_p[q,u,p]_\tau^\Box(t)\,, \\ \Box p(t) = -H_q[q,u,p]_\tau^\Box(t) - H_{q_\tau}[q,u,p]_\tau^\Box(t+\tau), \end{cases} \tag{3.17}$$

for $t_1 \le t \le t_2 - \tau$, *and by*

$$\begin{cases} \Box q(t) = H_p[q,u,p]_\tau^\Box(t)\,, \\ \Box p(t) = -H_q[q,u,p]_\tau^\Box(t)\,, \end{cases} \tag{3.18}$$

for $t_2 - \tau \le t \le t_2$; *the embedding of the stationary conditions* (3.15) *and* (3.16) *are given, respectively, by*

$$H_u[q,u,p]_\tau^\Box(t) + H_{u_\tau}[q,u,p]_\tau^\Box(t+\tau) = 0, \tag{3.19}$$

for $t_1 \le t \le t_2 - \tau$, *and by*

$$H_u[q,u,p]_\tau^\Box(t) = 0\,, \tag{3.20}$$

for $t_2 - \tau \le t \le t_2$.

Definition 3.14. We call systems (3.17) and (3.18) *the scale Hamiltonian systems with delay*, while to (3.19) and (3.20) we call *stationary conditions with delay*.

Lemma 3.15. *The embedding of* (3.10)–(3.11) *is given by*

$$I_\Box^\tau[q,u] = \int_{t_1}^{t_2} L\left(t, q(t), q(t-\tau), u(t), u(t-\tau)\right) dt \longrightarrow \min\,, \tag{3.21}$$

$$\Box q(t) = \varphi\left(t, q(t), q(t-\tau), u(t), u(t-\tau)\right)\,, \tag{3.22}$$

where $q, q\circ\rho^\tau \in H^\alpha\left(I, \mathbb{R}^d\right)$ *and* $u, u\circ\rho^\tau \in H^\alpha\left(I, \mathbb{C}^d\right)$.

Theorem 3.16 generalizes Theorem 3.12 for non-differentiable optimal control problems with time delay.

Theorem 3.16. *Let* $0 < \alpha, \beta < 1$ *with* $\alpha + \beta > 1$. *Assume that* $q \in H^{\alpha}\left(I, \mathbb{R}^d\right)$ *satisfies* $\Box q \in H^{\alpha}\left(I, \mathbb{R}^d\right)$ *and* $(L_{\Box q}[q]^{\tau}_{\Box}(t) + L_{\Box q_\tau}[q]^{\tau}_{\Box}(t)) \cdot h(t)$ *satisfies condition* (2.3) *for all* $h \in H^{\beta}\left(I, \mathbb{R}^d\right)$. *If* (q, u) *is a minimizer to problem* (3.21)–(3.22) *subject to given boundary conditions* (3.12), *then there exists a co-vector function* $p \in H^{\alpha}\left(I, \mathbb{C}^d\right)$ *such that the following conditions hold:*

- *the scale Hamiltonian systems with delay* (3.17) *and* (3.18)*;*
- *the stationary conditions with delay* (3.19) *and* (3.20).

Proof. We prove the theorem only in the interval $t_1 \leq t \leq t_2 - \tau$ (the reasoning is similar in interval $t_2 - \tau \leq t \leq t_2$). Using the Lagrange multiplier rule, (3.21)–(3.22) is equivalent to minimize the augmented functional $J^{\tau}_{\Box}[q, u, p]$ defined by

$$J^{\tau}_{\Box}[q, u, p] = \int_{t_1}^{t_2} \left[H\left(t, q(t), q(t-\tau), u(t), u(t-\tau), p(t)\right) - p(t) \cdot \Box q(t)\right] dt. \quad (3.23)$$

The necessary optimality conditions (3.17) and (3.19) are obtained from the Euler–Lagrange equations ($EL_{\Box LAP}$) applied to functional (3.23):

$$\begin{cases} \dfrac{\Box}{\Box t}\left(\mathbb{L}_{\Box q}[q,u,p]^{\tau}_{\Box}(t) + \mathbb{L}_{\Box q_\tau}[q,u,p]^{\tau}_{\Box}(t+\tau)\right) \\ \qquad\qquad = \mathbb{L}_{q}[q,u,p]^{\tau}_{\Box}(t) + \mathbb{L}_{q_\tau}[q,u,p]^{\tau}_{\Box}(t+\tau) \\ \dfrac{\Box}{\Box t}\left(\mathbb{L}_{\Box u}[q,u,p]^{\tau}_{\Box}(t) + \mathbb{L}_{\Box u_\tau}[q,u,p]^{\tau}_{\Box}(t+\tau)\right) \\ \qquad\qquad = \mathbb{L}_{u}[q,u,p]^{\tau}_{\Box}(t) + \mathbb{L}_{u_\tau}[q,u,p]^{\tau}_{\Box}(t+\tau) \\ \dfrac{\Box}{\Box t}\left(\mathbb{L}_{\Box p}[q,u,p]^{\tau}_{\Box}(t) + \mathbb{L}_{\Box p_\tau}[q,u,p]^{\tau}_{\Box}(t+\tau)\right) \\ \qquad\qquad = \mathbb{L}_{p}[q,u,p]^{\tau}_{\Box}(t) + \mathbb{L}_{p_\tau}[q,u,p]^{\tau}_{\Box}(t+\tau) \end{cases}$$

$$\Leftrightarrow \begin{cases} \Box p(t) = -H_q[q,u,p]^{\tau}_{\Box}(t) - H_{q_\tau}[q,u,p]^{\tau}_{\Box}(t+\tau) \\ 0 = H_u[q,u,p]^{\tau}_{\Box}(t) + H_{u_\tau}[q,u,p]^{\tau}_{\Box}(t+\tau) \\ 0 = -\Box q(t) + H_p[q,u,p]^{\tau}_{\Box}(t), \end{cases}$$

where $\mathbb{L}[q,u,p]^{\tau}_{\Box}(t) = H[q,u,p]^{\tau}_{\Box}(t) - p(t) \cdot \Box q(t)$. □

Remark 3.17. In the differentiable case Theorem 3.16 reduces to Theorem 3.12.

Remark 3.18. The first equations in the scale Hamiltonian system with delay (3.17) and (3.18) are nothing else than the scale control system (3.22).

Remark 3.19. In classical mechanics, p is called the *generalized momentum*. In the language of optimal control, p is known as the adjoint variable [16].

Definition 3.20. A triplet (q, u, p) satisfying the conditions of Theorem 3.16 will be called a *scale Pontryagin extremal*.

Remark 3.21. If $\varphi(t, q, q_\tau, u, u_\tau) = u$, then Theorem 3.16 reduces to Theorem 3.10. Let us verify this in the interval $t_1 \leq t \leq t_2 - \tau$ (the procedure is similar for $t_2 - \tau \leq t \leq t_2$). The stationary condition (3.19) gives $p(t) = L_u[q]_\tau^\Box(t) + L_{u_\tau}[q]_\tau^\Box(t+\tau)$ and the second equation in the scale Hamiltonian system with delay (3.17) gives $\Box p(t) = L_q[q]_\tau^\Box(t) + L_{q_\tau}[q]_\tau^\Box(t)$. Comparing both equalities, one obtains the non-differentiable Euler–Lagrange equations with time delay ($EL_{\Box LAP}$). In other words, the scale Pontryagin extremals (Definition 3.20) are a generalization of the non-differentiable Euler–Lagrange extremals (Definition 3.9).

We conclude from Theorem 3.16 that the coherence principle is also respected for non-differentiable optimal control problems with time delay.

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